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Random Walk on a Disordered Directed Bethe Lattice

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The random walk of a particle on a directed Bethe lattice of constant coordinance Z is examined in the case of random hopping rates. As a result, the higher the coordinance, the narrower the regions of anomalous drift and diffusion. The annealed and quenched mean square dispersions are calculated in all dynamical phases. In opposition to the one-dimensional ($Z=2$) case, the annealed and quenched mean quadratic dispersions are shown to be identical in all phases.

KEY WORDS: Fluctuation phenomena; random walks; disordered media.

1. INTRODUCTION

The properties of the random walk of a particle on a directed chain with random hopping rates are now well known.[1-3] This model is completely solvable analytically. The main reason for this lies in the fact that there exists in such a lattice a kind of order relation between successively visited sites. Indeed, in a one-dimensional directed lattice, a particle located initially at site 0 can only go from site 0 to site 1, and later from site n to its right nearest-neighbor site $n+1$. One thus can easily define a filiation relation between sites, each site having one "son" and one "father."

A similar filiation relation does exist on a directed Bethe lattice or infinite Cayley tree.[4] The directed Bethe lattice is defined as follows: in this

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[4] We shall employ indifferently the expressions Bethe lattice or infinite Cayley tree to denote an infinite ramified lattice of constant coordinance Z.[4,5]

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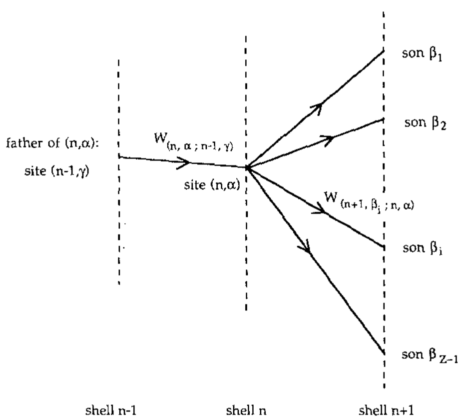
lattice, a particle located initially at site 0 can only go from site 0 to one of its $Z-1$ nearest neighbors, and later from site n, α to one of the $Z-1$ sites $n+1$, $\beta_i(\alpha)$. [The index n is a shell index and n, α denotes any site of this shell. The index $\beta_i(\alpha)$ varies from 0 to $Z-1$.] The nearest neighbors of site 0 form the $n=1$ shell, the second-nearest neighbors of site 0 form the $n=2$ shell,.... In this directed lattice also, one can define a filiation relation between sites, each site having $Z-1$ "sons" and one "father." The existence of this filiation relation is another aspect of the absence of closed cycles on the lattice. The case $Z=2$ is just the one-dimensional chain (see Fig. 1).

Note that the coordinance of the Bethe lattice is Z everywhere, except at the origin, where it is $Z-1$. In graph terminology, one is considering a rooted infinite Cayley tree.[6] Site 0 constitutes a kind of defect in the lattice, since it has no "father." However, in a *directed* Bethe lattice, any site has $Z-1$ "sons" and site 0 is no exception in this respect.

In the present paper, we shall be concerned with the problem of the random walk of a particle on such a directed Bethe lattice with random hopping rates. The techniques which allowed for the treatment on a linear directed chain are still applicable and actually lead to a complete analytical solution.

The paper is organized as follows. In Section 2, we describe the model and solve it in the case of nonrandom hopping rates. In Section 3, we consider random hopping rates and calculate the average over disorder of the mean particle position. In Section 4, we calculate the so-called "quenched"



$$W^s_{n,\alpha} = \sum_{i=1}^{Z-1} W_{(n+1,\beta_i;n,\alpha)}$$

Fig. 1. Schematic picture of the filiation in the directed Bethe lattice. We have indicated the notations used in the text.

and "annealed"[5] (as defined in ref. 7) mean quadratic dispersions of the particle, the former quantity being much more difficult to obtain. As a result, the higher the coordinance, the narrower the regions of anomalous drift and diffusion. In contradistinction to the one-dimensional case, the annealed (or average) and quenched mean quadratic dispersions are identical in all dynamical phases.

2. THE MODEL AND ITS SOLUTION IN THE ORDERED CASE

We consider the master equation describing the random walk of a particle on the directed Bethe lattice. If $p_{n,\alpha}(t)$ denotes the probability for the particle to be at site labeled n, α at time $t \geqslant 0$, one has

$$\frac{dp_{n,\alpha}}{dt} = W_{[n,\alpha;n-1,\gamma]}\, p_{n-1,\gamma}(t) - \sum_{i=1}^{Z-1} W_{[n+1,\beta_i;n,\alpha]}\, p_{n,\alpha}(t), \qquad n \geqslant 1 \tag{1}$$

In the above equation, $W_{[n+1,\beta_i;n,\alpha]}$ denotes the hopping rate from site n, α to one of its $Z-1$ "sons" and $W_{[n,\alpha;n-1,\gamma]}$ denotes the hopping rate from the "father" of site n, α toward this latter site. Note that the "father" $n-1, \gamma$ of site n, α is defined in a unique way. Moreover, we have dropped for brevity the explicit dependence of γ and β_i upon α. At time $t=0$ the particle is assumed to be located on site $n=0$. This site clearly has no "father" and $Z-1$ "sons." For it, the master equation simply reads

$$\frac{dp_0}{dt} = -\sum_{i=1}^{Z-1} W_{[1,\beta_i;0]}\, p_0(t) \tag{2}$$

Let us first assume that all the hopping rates take the same nonrandom value W. In such a case, the probability for the particle to be at a given site only depends on the shell index n.

As usual in random walk problems, one performs a Laplace transformation of the master equations (1) and (2), which yields

$$zP_n(z) = WP_{n-1}(z) - (Z-1)\, WP_n(z), \qquad n \geqslant 1 \tag{3}$$

and

$$zP_0(z) - 1 = -(Z-1)\, WP_0(z) \tag{4}$$

Equations (3) and (4) are trivially solvable. One gets

$$P_n(z) = \frac{1}{z + (Z-1)\, W} \left(\frac{W}{z + (Z-1)\, W} \right)^n, \qquad n \geqslant 0 \tag{5}$$

[5] Note that the same quantity is called "average" quadratic dispersion in ref. 1. In the following, we shall refer to this quantity as to the "annealed dispersion."

These probabilities are correctly normalized. Indeed, since the shell of index n contains $(Z-1)^n$ sites, one easily verifies that

$$\sum_{n=0}^{\infty} (Z-1)^n P_n(z) = \frac{1}{z} \tag{6}$$

The main physical quantities of interest in the random walk are the mean particle position and its mean quadratic dispersion. The mean particle position actually refers to the mean shell index, that is, to the mean number of steps that the particle has done since its departure from the origin. The actual number of sites in a given shell has to be taken into account, so that the mean particle position is defined as follows:

$$\overline{x(t)} = \sum_{n=0}^{\infty} n(Z-1)^n p_n(t) \tag{7}$$

In a similar way, one can define the mean second moment of the number of steps from the origin

$$\overline{x^2(t)} = \sum_{n=0}^{\infty} n^2(Z-1)^n p_n(t) \tag{8}$$

The mean quadratic dispersion of the particle is defined as

$$\overline{\Delta x^2(t)} = \overline{x^2(t)} - \overline{x(t)}^2 \tag{9}$$

The Laplace transforms of the mean particle position and of its mean second moment, $x_1(z)$ and $x_2(z)$, are easily obtained from Eq. (5). One gets

$$x_1(z) = \frac{W(Z-1)}{z^2}, \qquad x_2(z) = 2\,\frac{[W(Z-1)]^2}{z^3} + \frac{W(Z-1)}{z^2} \tag{10}$$

from which one derives the time-dependent quantities

$$\overline{x(t)} = Vt, \qquad \overline{\Delta x^2(t)} = W(Z-1)\,t \tag{11}$$

The regime is a normal drift-diffusion one, with a velocity V given by

$$V = W(Z-1) \tag{12}$$

and a diffusion coefficient given by

$$D = \frac{V}{2} = \frac{W(Z-1)}{2} \tag{13}$$

This result was to be expected, since the $Z-1$ diffusion channels on each site are independent. Note that for $Z=2$ the results for the one-dimensional case are trivially recovered.

3. THE DISORDERED DIRECTED BETHE LATTICE

We now solve the random walk equations (1) and (2) when the hopping rates are chosen independently at random in a given probability law $\rho(W)$. We choose a gamma probability distribution of parameter μ

$$\rho(W)\,dW=\frac{1}{\Gamma(\mu)}\left(\frac{W}{W_c}\right)^{\mu-1} e^{-W/W_c}\frac{dW}{W_c} \qquad (W>0, \quad \mu>0) \tag{14}$$

Here W_c denotes a fixed cutoff frequency.[6] The smaller μ, the higher the probability to find a quasibroken link.[2,3] Therefore, one expects a slowing down of the motion when μ is decreased toward 0_+, as in the corresponding one-dimensional situation. Indeed, decreasing the single parameter μ allows one to go from weak to strong disorder.

For each configuration of disorder, one can consider the probability for the particle to be at a given site as well as its mean position and its mean quadratic dispersion. These latter quantities will be denoted by the same symbol $\overline{\cdots}$ as in the preceding section. However, in the disordered case, these quantities are configuration-dependent and difficult to characterize. Thus we shall give the corresponding averages over disorder and analyze in due time the self-averaging properties. The average over disorder will be denoted in the following by the symbol $\langle\cdots\rangle$.

As in the ordered case, one performs a Laplace transformation of the master equations (1) and (2), which now yields

$$zP_{n,\alpha}(z)=W_{[n,\alpha;n-1,\gamma]}P_{n-1,\gamma}(z)-\sum_{i=1}^{Z-1} W_{[n+1,\beta_i;n,\alpha]}P_{n,\alpha}(z), \qquad n\geqslant 1 \tag{15}$$

and

$$zP_0(z)-1=-\sum_{i=1}^{Z-1} W_{[1,\beta_i;0]}P_0(z) \tag{16}$$

Equation (16) is still easy to handle, since it is a closed equation for $P_0(z)$. One gets

$$P_0(z)=\frac{1}{z+\sum_{i=1}^{Z-1} W_{[1,\beta_i;0]}} \tag{17}$$

[6] The cutoff function has been taken to be of an exponential form. On physical grounds, it is expected that the results at large times should be independent of the particular choice of this function.

3.1. The Average Probabilities

A useful quantity is the average over disorder of $P_0(z)$

$$\langle P_0(z)\rangle = \left\langle \frac{1}{z+\sum_{i=1}^{Z-1} W_{[1,\beta_i;0]}} \right\rangle \tag{18}$$

A random quantity of central interest in this problem turns out to be the sum of all the hopping rates which link together the origin and its $Z-1$ "sons," that is,

$$W_0^s = \sum_{i=1}^{Z-1} W_{[1,\beta_i;0]} \tag{19}$$

An analogous quantity $W_{n,\alpha}^s$ can be defined for each site n,α. In the following, we shall denote it W^s for short, except when the explicit site dependence will be needed for clarity.

By rewriting

$$\langle P_0(z)\rangle = \left\langle \frac{1}{z+W^s} \right\rangle \tag{20}$$

one obtains a formula analogous to the corresponding one on a one-dimensional directed lattice,[2,3] except for the replacement of W by W^s.

Since the gamma distribution is stable for the addition of random variables, W^s is also distributed according to a gamma distribution, with the parameter $m=(Z-1)\,\mu$,

$$\rho^s(W^s)\,dW^s = \frac{1}{\Gamma(m)} \left(\frac{W^s}{W_c}\right)^{m-1} e^{-W^s/W_c} \frac{dW^s}{W_c} \qquad (W^s>0, \quad m>0) \tag{21}$$

The Laplace transform of the probability for the particle to be at the origin will have the same expression as on a directed one-dimensional lattice, except for the replacement of μ by $m=(Z-1)\,\mu$. One gets

$$\langle P_0(z)\rangle = \frac{z^{m-1}}{W_c^m} e^{z/W_c} \Gamma\left(1-m, \frac{z}{W_c}\right) \tag{22}$$

where Γ is the incomplete gamma function. This yields the following asymptotic time behavior

$$\langle P_0(t)\rangle \sim (W_c t)^{-m}, \qquad t\to\infty \tag{23}$$

The Laplace transforms of the other probabilities of presence can then be calculated step by step. For this purpose, it may be useful to rewrite the master equations (15) and (16) in the following way:

$$zP_{n,\alpha}(z) = W_{[n,\alpha;n-1,\gamma]}P_{n-1,\gamma}(z) - W^s_{n,\alpha}P_{n,\alpha}(z), \qquad n \geqslant 1 \tag{24}$$

$$zP_0(z) - 1 = -W^s_0 P_0(z) \tag{25}$$

For any given configuration of the hopping rates, one can check that the probabilities are normalized, as required. Indeed, summing over the various sites gives, from Eqs. (24) and (25),

$$z \sum_{n=0}^{\infty} \sum_{\alpha=1}^{(Z-1)^n} P_{n,\alpha}(z) - 1 = \sum_{n=1}^{\infty} \sum_{\gamma=1}^{(Z-1)^{n-1}} W^s_{n-1,\gamma}P_{n-1,\gamma}(z) - \sum_{n=0}^{\infty} \sum_{\alpha=1}^{(Z-1)^n} W^s_{n,\alpha}P_{n,\alpha}(z) \tag{26}$$

which is obviously equal to zero. In Eq. (26), the sum on all sites of the shell of index n of the products of the hopping rates $W_{[n,\alpha;n-1,\gamma]}$ by the probabilities $P_{n-1,\gamma}(z)$ has been rewritten as a sum on all sites of the shell of index $n-1$ of the products of the hopping rates $W^s_{n-1,\gamma}$ by the probabilities $P_{n-1,\gamma}(z)$.

Before computing any average values, let us note that Eq. (24) displays that a site n, α is linked to its "father" $n-1, \gamma$ via a hopping rate distributed according to the law (14), while it is linked to its "sons" $n+1, \beta_i$ via a hopping rate distributed according to the law (21). The average values $\langle P_{n,\alpha}(z)\rangle$ are actually independent of α and can be easily computed. One gets

$$\langle P_n(z)\rangle = \left\langle \frac{1}{z+W^s}\right\rangle \left\langle \frac{W}{z+W^s}\right\rangle^n \tag{27}$$

Here W is one of the statistically independent $Z-1$ "components" of W^s, so that

$$\left\langle \frac{W}{z+W^s}\right\rangle = \frac{1}{Z-1}\left\langle \frac{W^s}{z+W^s}\right\rangle \tag{28}$$

3.2. The Average over Disorder of the Mean Particle Position

We now calculate the average over disorder of the mean particle position, as defined by

$$\langle \overline{x(t)}\rangle = \sum_{n=0}^{\infty} n(Z-1)^n \langle p_{n,\alpha}(t)\rangle \tag{29}$$

Using Eq. (27), one obtains the Laplace transform $\langle x_1(z) \rangle$ of this quantity,

$$\langle x_1(z) \rangle = \frac{1}{z^2 \langle 1/(z+W^s) \rangle} - \frac{1}{z} \tag{30}$$

The asymptotic behavior of $\langle \overline{x(t)} \rangle$ is easily obtained. When $m = (Z-1)\mu$ is smaller than 1, one has an anomalous drift behavior, characterized by

$$\langle \overline{x(t)} \rangle \sim \frac{\sin \pi m}{\pi m} (W_c t)^m, \qquad 0 < m < 1 \tag{31}$$

When m is larger than 1, the drift regime is normal, with a finite velocity

$$\langle \overline{x(t)} \rangle \sim (m-1) W_c t, \qquad m > 1 \tag{32}$$

As far as the position is concerned, one recovers the one-dimensional behavior, with μ changed into $(Z-1)\mu$. In this respect, the Bethe lattice may be considered as quasi-one-dimensional. This can be traced back to the fact that there is only one way to go from the origin to any other site. For a given μ, the width of the anomalous phase decreases as the coordinance Z increases.

4. THE MEAN QUADRATIC DISPERSION OF THE PARTICLE

The average over disorder of the mean quadratic dispersion of the particle may be defined in two different ways. One can define both the quenched dispersion

$$\langle \overline{\Delta x^2(t)} \rangle_Q = \langle \overline{x^2(t)} \rangle - \langle \overline{x(t)}^2 \rangle \tag{33}$$

and the annealed one

$$\langle \overline{\Delta x^2(t)} \rangle_A = \langle \overline{x^2(t)} \rangle - \langle \overline{x(t)} \rangle^2 \tag{34}$$

The knowledge of the average probabilities is sufficient to obtain the annealed dispersion, but *not* the quenched one. While the annealed dispersion characterizes the spread of a configuration-averaged packet, the quenched dispersion characterizes the spread of a packet in a single environment, the average over disorder being only taken at the end of the calculation.[7]

4.1. Why Two Different Definitions of the Mean Quadratic Dispersion

Clearly, the above question only makes sense in a disordered medium. In the normal diffusion regime, the reason which allows $\langle \overline{\Delta x^2(t)} \rangle_A$

to be different from $\langle\overline{\Delta x^2(t)}\rangle_Q$ is the sample-to-sample fluctuation of the correction to the leading behavior of the mean particle position $\overline{x(t)}$. It is expected that, when diffusion is normal, this fluctuation exists only in one dimension and when a global bias is present, while, in more than one dimension, $\langle\overline{\Delta x^2(t)}\rangle_A = \langle\overline{\Delta x^2(t)}\rangle_Q$.[7]

When diffusion is anomalous, the spread for a given configuration of disorder and the average spread can have a very different behavior in the presence of a bias.[7]

The present model, since it is directed, is clearly biased. However, the question of the effective "dimension" of the Bethe lattice is not so clear. The answer may depend on the particular property under study.[4,5,8] We have shown in Section 3 that the average over disorder of the mean particle position behaves like the corresponding one-dimensional quantity, except for the replacement of the parameter μ characterizing the distribution of the hopping rates by $m = (Z-1)\,\mu$. The replacement of μ by m leads to a reduced width of the anomalous phase of drift as compared to the one-dimensional case. One could then be led to think of the directed Bethe lattice as one-dimensional for the above-quoted property. But, where this lattice truly one-dimensional, one could expect different behaviors for the annealed and quenched mean quadratic dispersions of the particle. As a matter of fact, we shall show that this is *not* the case. In the directed Bethe lattice, $\langle\overline{\Delta x^2(t)}\rangle_Q$ and $\langle\overline{\Delta x^2(t)}\rangle_A$ are actually identical in all dynamical phases, anomalous or not. This identity in the normal regime of $\langle\overline{\Delta x^2(t)}\rangle_Q$ and of $\langle\overline{\Delta x^2(t)}\rangle_A$ is a property also expected for higher-dimensional Euclidean lattices.[7],[7]

4.2. The Annealed Average over Disorder of the Mean Quadratic Dispersion of the Particle

One first calculates the average over disorder of the mean second moment of the particle position, as defined by

$$\langle\overline{x^2(t)}\rangle = \sum_{n=0}^{\infty} n^2 (Z-1)^n \langle P_n(t)\rangle \tag{35}$$

Using Eq. (27), one obtains the Laplace transform $\langle x_2(z)\rangle$ of this quantity,

$$\langle x_2(z)\rangle = \frac{2}{z^3\langle 1/(z+W^s)\rangle^2} - \frac{3}{z^2\langle 1/(z+W^s)\rangle} + \frac{1}{z} \tag{36}$$

[7] In a recent calculation,[9] we have shown that this identity is true in any phase for a directed walk on a d-dimensional ($d \geqslant 2$) Euclidean lattice.

Combining the asymptotic behaviors of $\langle\overline{x^2(t)}\rangle$ [deduced from Eq. (36)] and of $\langle\overline{x(t)}\rangle$ [deduced from Eq. (30)], one gets the asymptotic behavior of $\langle\overline{\Delta x^2(t)}\rangle_A$. The results only depend on the value of $m=(Z-1)\,\mu$. When m is smaller than 1, one has an anomalous diffusion behavior, characterized by

$$\langle\overline{\Delta x^2(t)}\rangle_A \sim \frac{1}{\Gamma(2m+1)[\Gamma(1-m)]^2}\left[2-\frac{\Gamma(2m+1)}{[\Gamma(m+1)]^2}\right](W_c t)^{2m} \tag{37}$$

When $1<m<2$, one also has an anomalous diffusion behavior, but characterized by

$$\langle\overline{\Delta x^2(t)}\rangle_A \sim 2\,\frac{(m-1)^3}{(2-m)(3-m)}\,(W_c t)^{3-m} \tag{38}$$

When m is larger than 2, the diffusion regime is normal, with a finite diffusion coefficient

$$\langle\overline{\Delta x^2(t)}\rangle_A \sim \frac{m(m-1)}{m-2}\,W_c t \tag{39}$$

The conclusion for the diffusion is the same as for the velocity: here also one recovers a one-dimensional-like behavior. For a given μ, the width of the anomalous phase decreases as the coordinance Z increases.

4.3. The Functional Relation for $x_1(z)$

As in the one-dimensional directed lattice, a useful tool for calculating $\langle\overline{\Delta x^2(t)}\rangle_Q$ is the so-called functional relation obeyed by $x_1(z)$.[2,3] For any given configuration of the lattice

$$x_1(z)=\sum_{n=0}^{\infty} n \sum_{\alpha=1}^{(Z-1)^n} P_{n,\alpha}(z) \tag{40}$$

In the master equation (15), the sum on all sites of the shell of index n of the products of the hopping rates $W_{[n,\alpha;n-1,\gamma]}$ by the probabilities $P_{n-1,\gamma}(z)$ can be rewritten as a sum on all sites of the shell of index $n-1$ of the products of the hopping rates $W^s_{n-1,\gamma}$ by the probabilities $P_{n-1,\gamma}(z)$. By taking into account the resulting equations (24) and (25), and using the same trick as in deriving Eq. (26), one gets

$$\begin{aligned} zx_1(z) = {} & \sum_{n=1}^{\infty} n \sum_{\gamma=1}^{(Z-1)^{n-1}} W^s_{n-1,\gamma} P_{n-1,\gamma}(z) \\ & - \sum_{n=0}^{\infty} n \sum_{\alpha=1}^{(Z-1)^n} W^s_{n,\alpha} P_{n,\alpha}(z) \end{aligned} \tag{41}$$

or simply

$$zx_1(z) = \sum_{n=0}^{\infty} \sum_{\alpha=1}^{(Z-1)^n} W^s_{n,\alpha} P_{n,\alpha}(z) \tag{42}$$

By singling out the term corresponding to the first site, one can rewrite Eq. (42) under the form of a functional relation,

$$zx_1(0; z) = \frac{W^s_0}{z + W^s_0} \left[1 + \sum_{\alpha=1}^{Z-1} \frac{W_{[1,\alpha;0]}}{W^s_0} zx_1(1, \alpha; z) \right] \tag{43}$$

Note that, in the above equation, $x_1(0; z)$ is calculated for a particle which, at $t = 0$, is located on site $n = 0$, while $x_1(1, \alpha; z)$ is calculated for a particle which, at $t = 0$, is located on site $1, \alpha$. Since site 0 constitutes no exception in the directed Bethe lattice as long as the "sons" are considered, these two quantities will be equivalent once averages over disorder are taken.

The quenched average $\langle \overline{\Delta x^2(t)} \rangle_Q$ can be obtained by a Laplace inversion of the quantity

$$\langle \Delta_2(z) \rangle = \langle x_2(z) - (x_1 * x_1)(z) \rangle \tag{44}$$

where the symbol $*$ denotes the convolution product.

As in the one-dimensional case, the functional relation will serve as a basis for calculating the average over disorder of the product $x_1(z)\, x_1(z')$.

One can remark that this functional relation presents some analogies with the recursion relation obtained in refs. 10 and 11 for the partition function of a directed polymer on a tree. These analogies are of course based on the underlying tree structure in both problems.

4.4. Calculation of $\langle x_1(z)\, x_1(z') \rangle$

By applying twice the functional relation (43), it is actually possible to calculate $\langle x_1(z)\, x_1(z') \rangle$. One begins by writing

$$\begin{aligned} zz'x_1(0; z)\, x_1(0; z') = {} & \frac{W^s_0}{z + W^s_0} \frac{W^s_0}{z' + W^s_0} \left[1 + \sum_{\alpha=1}^{Z-1} \frac{W_{[1,\alpha;0]}}{W^s_0} zx_1(1, \alpha; z) \right] \\ & \times \left[1 + \sum_{\beta=1}^{Z-1} \frac{W_{[1,\beta;0]}}{W^s_0} z'x_1(1, \beta; z') \right] \end{aligned} \tag{45}$$

At this stage, the calculation in the Bethe lattice differs from the one in the one-dimensional directed chain. When α is not equal to β, the two quantities $x_1(1, \alpha; z)$ and $x_1(1, \beta; z')$ are uncorrelated. This direct effect of the branching of the lattice clearly could not happen in one dimension. By taking the configuration average of Eq. (45), one gets

$$\begin{aligned}
&zz'\langle x_1(0; z)\, x_1(0; z')\rangle \\
&= \left\langle \frac{W^s}{z+W^s}\frac{W^s}{z'+W^s}\right\rangle \\
&+ z\langle x_1(z)\rangle(Z-1)\left\langle \frac{W}{z+W_s}\frac{W^s}{z'+W^s}\right\rangle \\
&+ z'\langle x_1(z')\rangle\langle Z-1)\left\langle \frac{W}{z'+W_s}\frac{W^s}{z+W^s}\right\rangle \\
&+ zz'\langle x_1(1,\alpha; z)\, x_1(1,\beta; z')\rangle \sum_{\alpha=1}^{Z-1}\sum_{\beta=1}^{Z-1}\left\langle \frac{W_{[1,\alpha;0]}}{z+W_0^s}\frac{W_{[1,\beta;0]}}{z'+W_0^s}\right\rangle
\end{aligned} \tag{46}$$

In the above equation, some indexes have been dropped when there was no ambiguity. W denotes one of the $Z-1$ "components" of W^s.

Let us now successively examine the different averaged quantities which are involved in Eq. (46). First, one easily shows that

$$\left\langle \frac{W^s}{z+W^s}\frac{W^s}{z'+W^s}\right\rangle = 1 - \frac{z^2R(z)-z'^2R(z')}{z-z'} \tag{47}$$

where for short

$$R(z) \equiv \left\langle \frac{1}{z+W^s}\right\rangle \tag{48}$$

In the same way, taking into account Eq. (28), one shows that the second term in the right-hand side of Eq. (46) is given by

$$z\langle x_1(z)\rangle\left[1-\frac{z^2R(z)-z'^2R(z')}{z-z'}\right] \tag{49}$$

or, using expression (30) for $\langle x_1(z)\rangle$,

$$\left[\frac{1}{zR(z)}-1\right]\left[1-\frac{z^2R(z)-z'^2R(z')}{z-z'}\right] \tag{50}$$

The third term of the right-hand side of Eq. (45) is obtained by changing z into z' in Eq. (50). Finally, the fourth term of Eq. (46) has to be carefully analyzed, since it brings in the effect of the branching of the lattice. This sum of $(Z-1)^2$ terms contains $Z-1$ "diagonal" terms (with $\alpha=\beta$) and $(Z-1)(Z-2)$ "off-diagonal" terms (with $\alpha\neq\beta$). If W and W' denote two different "components" of W^s, the two averages

$\langle W^2/[(z+W^s)(z'+W^s)]\rangle$ and $\langle WW'/[(z+W^s)(z'+W^s)]\rangle$ have to be computed. One can show that a ratio independent of z does exist between these two quantities. For instance, with the gamma distribution laws (14) for W and W' and (21) for W^s, one has

$$\left\langle\frac{W^2}{(z+W^s)(z'+W^s)}\right\rangle=\frac{\mu+1}{\mu}\left\langle\frac{WW'}{(z+W^s)(z'+W^s)}\right\rangle \tag{51}$$

(details are provided in the Appendix). As a result

$$\left\langle\frac{W^2}{(z+W^s)(z'+W^s)}\right\rangle=\frac{1}{(Z-1)(1+K)}\left[1-\frac{z^2R(z)-z'^2R(z')}{z-z'}\right] \tag{52}$$

$$\left\langle\frac{WW'}{(z+W^s)(z'+W^s)}\right\rangle=\frac{\mu}{\mu+1}\frac{1}{(Z-1)(1+K)}\times\left[1-\frac{z^2R(z)-z'^2R(z')}{z-z'}\right] \tag{53}$$

where we have set

$$K=(Z-2)\frac{\mu}{\mu+1} \tag{54}$$

It clearly appears from the Appendix that the explicit form of K depends on the choice of the cutoff function. However, it is readily shown that, for any choice of $\rho(W)$, K could be a function of z, which does not vanish for $z\to 0$, as well as $1+K$. The fourth term in Eq. (46) is thus given by

$$zz'\langle x_1(1,\alpha;z)\,x_1(1,\alpha;z')\rangle\frac{1}{1+K}\left[1-\frac{z^2R(z)-z'^2R(z')}{z-z'}\right]+\frac{K}{1+K}\left[\frac{1}{zR(z)}-1\right]\left[\frac{1}{z'R(z')}-1\right]\left[1-\frac{z^2R(z)-z'^2R(z')}{z-z'}\right] \tag{55}$$

Since the two averages $\langle x_1(0;z)\,x_1(0;z')\rangle$ and $\langle x_1(1,\alpha;z)\,x_1(1,\alpha;z')\rangle$ are identical, one gets as a final expression to be analyzed for $\langle x_1(z)\,x_1(z')\rangle$

$$zz'\langle x_1(z)\,x_1(z')\rangle\left\{1-\frac{1}{1+K}\left[1-\frac{z^2R(z)-z'^2R(z')}{z-z'}\right]\right\}=\left[1-\frac{z^2R(z)-z'^2R(z')}{z-z'}\right]\left\{\frac{1}{zR(z)}+\frac{1}{z'R(z')}-1+\frac{K}{1+K}\left[\frac{1}{zR(z)}-1\right]\left[\frac{1}{z'R(z')}-1\right]\right\} \tag{56}$$

4.5. Calculation of the Quenched Average $\langle \overline{\Delta x^2(t)} \rangle_Q$

In order to obtain $\langle \overline{\Delta x^2(t)} \rangle_Q$, it remains first to derive from Eq. (56) the small-z behavior of $\langle (x_1 * x_1)(z) \rangle$, to combine it with the small-z behavior of $\langle x_2(z) \rangle$ [deduced from Eq. (36)], and finally to resort to inverse Laplace transformation. One is thus led to study the integral in the complex-z' plane of

$$\begin{aligned}
&\langle x_1(z')\, x_1(z-z') \rangle \\
&= -\frac{1}{z'(z-z')}\left[\frac{1}{z'R(z')} + \frac{1}{(z-z')\,R(z-z')} - 1 + \frac{K}{z'(z-z')\,R(z')\,R(z-z')}\right] \\
&\quad + \frac{1+K}{z'(z-z')}\left[\frac{1}{z'R(z')} + \frac{1}{(z-z')\,R(z-z')} - 1\right. \\
&\quad \left. + K\frac{1}{z'(z-z')\,R(z')\,R(z-z')}\right] \\
&\quad \times \left[K + \frac{z'^2R(z') - (z-z')^2\,R(z-z')}{2z'-z}\right]^{-1}
\end{aligned} \tag{57}$$

The integration line is parallel to the imaginary axis and chosen so as to ensure both $\operatorname{Re} z' > 0$ and $\operatorname{Re}(z-z') > 0$. We have now to resort to the conventional complex integration techniques. As usual, we deform the initial contour so as to include the cut of the multivalued function $R(z)$ defined along the semiaxis $\operatorname{Re} z \leqslant 0$.

The three cases $m < 1$, $1 < m < 2$, and $m > 2$ will have to be separately analyzed, since the two first terms at small z of $R(z)$ are different in these three cases. From formula (22), one gets

$$R(z) \sim \frac{\Gamma(1-m)}{W_c^m}\, z^{m-1}, \qquad m < 1 \tag{58}$$

$$R(z) \sim \frac{1}{(m-1)\,W_c} + \frac{\Gamma(1-m)}{W_c^m}\, z^{m-1}, \qquad 1 < m < 2 \tag{59}$$

$$R(z) \sim \frac{1}{(m-1)\,W_c} - \frac{1}{(m-1)(m-2)}\,\frac{z}{W_c^2}, \qquad m > 2 \tag{60}$$

The first term [let us call it $T_1(z)$ for short] in the right-hand side of Eq. (57) is very easily computed without resorting to complex integration, since it can be rewritten in the equivalent form

$$T_1(z) = (1+K)\left[\frac{1}{z} - \frac{2}{z^2R(z)}\right] - K\langle x_1 \rangle * \langle x_1 \rangle(z) \tag{61}$$

After Laplace inversion, this gives a contribution $T_1(t)$ to $\langle \overline{x(t)}^2 \rangle$,

$$T_1(t) \sim -\frac{K}{[\Gamma(1-m)]^2} \frac{(W_c t)^{2m}}{[\Gamma(m+1)]^2}, \qquad m<1 \tag{62}$$

$$T_1(t) \sim -K\left[(m-1)^2 (W_c t)^2 - 2\frac{(m-1)^3}{(2-m)(3-m)} (W_c t)^{3-m} \right], \qquad 1<m<2 \tag{63}$$

$$T_1(t) \sim -K(m-1)^2 (W_c t)^2 - 2(m-1) W_c t - 2K\frac{(m-1)^2}{m-2} W_c t, \quad m>2 \tag{64}$$

where we used the same symbol for the function and its Laplace transform.

Let us now examine the second term in the right-hand side of Eq. (57), which we call here $T_2(z)$ for short. Since K is a strictly positive constant in a Bethe lattice of coordinance higher than 2 (we exclude the one-dimensional chain), no pole tending toward zero with z can arise from the denominator

$$K + \frac{z'^2 R(z') - (z-z')^2 R(z-z')}{2z'-z}$$

whatever the value of m. The leading contributions thus come from the cut of the multivalued function $R(z)$. After a somewhat lengthy calculation, one gets the behavior at small z of $T_2(z)$ and, subsequently, after Laplace inversion,

$$T_2(t) \sim \frac{1+K}{[\Gamma(1-m)]^2} \frac{(W_c t)^{2m}}{[\Gamma(m+1)]^2}, \qquad m<1 \tag{65}$$

$$T_2(t) \sim (1+K)\left[(m-1)^2 (W_c t)^2 - 2\frac{(m-1)^3}{(2-m)(3-m)} (W_c t)^{3-m} \right], \quad 1<m<2 \tag{66}$$

$$T_2(t) \sim (1+K)(m-1)^2 (W_c t)^2 + 2(1+K)\frac{(m-1)^2}{(m-2)} W_c t, \qquad m>2 \tag{67}$$

Finally

$$\langle \overline{\Delta x^2(t)} \rangle_Q = \langle \overline{x^2(t)} \rangle - [T_1(t) + T_2(t)] \tag{68}$$

As a result, the quenched average of the mean quadratic dispersion of the particle does not involve the parameter K, the only relevant parameter being m. This was indeed expected, since the precise form of K depends on the cutoff function.

The results are as follows. When m is smaller than 1, the diffusion regime is anomalous and characterized by

$$\langle \overline{\Delta x^2()t)} \rangle_Q \sim \frac{1}{\Gamma(2m+1)[\Gamma(1-m)]^2}\left[2 - \frac{\Gamma(2m+1)}{[\Gamma(m+1)]^2}\right](W_c t)^{2m} \tag{69}$$

When $1 < m < 2$, one also has an anomalous diffusion behavior, but characterized by

$$\langle \overline{\Delta x^2(t)} \rangle_Q \sim 2\,\frac{(m-1)^3}{(2-m)(3-m)}\,(W_c t)^{3-m} \tag{70}$$

When m is larger than 2, the diffusion regime is normal, with a finite diffusion coefficient

$$\langle \overline{\Delta x^2(t)} \rangle_Q \sim \frac{m(m-1)}{m-2}\,W_c t \tag{71}$$

As a conclusion, the annealed and quenched averages of the mean square dispersion of the particle are identical in all dynamical phases, anomalous or not. Indeed the dominant terms in $\langle \overline{x(t)^2} \rangle$ are due to particles which take different trajectories on the Bethe lattice. In these dominant terms, the random variables of interest are therefore independent, which in turn ensures the identity between the annealed and quenched averages of the mean quadratic dispersion of the particle. As compared to what happens on a directed one-dimensional chain,[2,3] this is clearly an effect of the branching character of the Bethe lattice. The extension to a d-dimensional Euclidean lattice has been achieved in ref. 9.

Note that this result can also be obtained by directly considering the Laplace transform $\langle x_1(z')\, x_1(z-z') \rangle - \langle x_1(z') \rangle \langle x_1(z-z') \rangle$ of the quantity $\langle \overline{x(t)^2} \rangle - \langle \overline{x(t)} \rangle^2$ and showing that it leads to a subdominant contribution.

As a final remark, let us note that the identity between the annealed and quenched mean quadratic dispersions of the particle implies that the mean position $\overline{x(t)}$ is self-averaging at large times in all dynamical phases, in contradistinction to the case $Z = 2$.[2,3] In other words, in all phases and at large times, the difference $\langle \overline{x(t)^2} \rangle - \langle \overline{x(t)} \rangle^2$ is subdominant. One could ask whether the diffusion front itself is self-averaging. This would require the knowledge of the self-averaging properties of higher cumulants, which remains an open problem.

5. CONCLUSION

We have extended to a Bethe lattice of constant coordinance Z the directed model of a random walk. This was rendered possible by the filiation relation which exists between the sites of a directed Bethe lattice.

The results can be summarized as follows. First, the drift and diffusion properties of the particle present some analogies with the corresponding properties of a particle on a directed chain. When quasibroken links become sufficiently numerous, the motion of the particle is slowed down and phases of anomalous drift and diffusion appear. However, the higher the coordinance, the narrower the regions of anomalous drift and diffusion.

Second, and this is in marked opposition with the case of a one-dimensional lattice, the annealed and quenched mean quadratic dispersions of the particle are identical in all dynamical phases. This may be traced back to the fact that, for $Z>2$, the number of visited sites grows rapidly (here exponentially) with time, instead of linearly in the one-dimensional case.

Some more light can be shed on the importance of the branching of the lattice by considering the average over disorder of $\langle P_0(z)\rangle$. In a one-dimensional lattice, the two first terms at small z correspond in the normal drift and diffusion regime to an expansion of the form

$$\langle P_0(z)\rangle \sim \frac{1}{V} - \frac{2zD}{V^3} \tag{72}$$

where V and D are, respectively, the drift velocity and the quenched diffusion coefficient of the particle. With the choice (14) of the probability distribution of the hopping rates, one gets, in a one-dimensional lattice

$$V = (\mu - 1)\, W_c, \qquad D = \frac{(\mu-1)^2}{2(\mu-2)}\, W_c \tag{73}$$

If the same was true for a Bethe lattice, that is, if V and D could be extracted from $\langle P_0(z)\rangle$ alone, one would have

$$V = (m - 1)\, W_c, \qquad D = \frac{(m-1)^2}{2(m-2)}\, W_c \tag{74}$$

The value (74) for the velocity is the correct one [see formula (32)]. However, the value (74) for the diffusion coefficient is not exact, as displayed by formulas (39) and (71). This clearly shows that, when Z is higher than 2, it is no longer possible to extract the proper diffusion coefficient from the knowledge of $\langle P_0(z)\rangle$ alone. This most probably has to be related to the fact that a particle initially located at site 0 has many possible spatially different ways to achieve its trajectory. In particular, this is reflected by the fact that the value of D extracted from $\langle P_0(z)\rangle$ is smaller than the exact one. Finally, let us remark that, when $m \gg 2$, $D \simeq V/2$, as in the ordered case.

One might think that these effects of the branching may persist on higher-dimensional disordered Euclidean lattices. Actually, as noted above, we have shown that they persist for a directed walk on a d-dimensional ($d \geqslant 2$) Euclidean lattice.[9] A complete study of random walks in such lattices would obviously be much more involved, mainly due to the presence of closed cycles and of correlations difficult to take properly into account.

APPENDIX

Let us here demonstrate that a ratio independent of z does exist between the two averages $\langle W^2/[(z+W^s)(z'+W^s)]\rangle$ and $\langle WW'/[(z+W^s)(z'+W^s)]\rangle$, where W and W' denote two different "components" of W^s. First, let us rewrite these quantities as

$$\left\langle\frac{W^s}{(z+W^s)(z'+W^s)}\right\rangle=\frac{1}{z'-z}\left[\left\langle\frac{W^2}{z+W^s}\right\rangle-\left\langle\frac{W^2}{z'+W^s}\right\rangle\right] \tag{A1}$$

and

$$\left\langle\frac{WW'}{(z+W^s)(z'+W^s)}\right\rangle=\frac{1}{z'-z}\left[\left\langle\frac{WW'}{z+W^s}\right\rangle-\left\langle\frac{WW'}{z'+W^s}\right\rangle\right] \tag{A2}$$

One has

$$\begin{aligned}\left\langle\frac{W^2}{z+W^s}\right\rangle &\equiv \left\langle\frac{W_i^2}{z+W_1+\cdots+W_{Z-1}}\right\rangle\\ &=\frac{W_c}{[\Gamma(\mu)]^{Z-1}}\int_0^\infty\cdots\int_0^\infty dx_1\cdots dx_{Z-1}\\ &\quad\times\frac{x_i^2x_1^{\mu-1}\cdots x_{Z-1}^{\mu-1}}{z/W_c+x_1+\cdots+x_{Z-1}}\,e^{-(x_1+\cdots+x_{Z-1})}\end{aligned} \tag{A3}$$

where x_i stands for the ratio W_i/W_c. This multiple integral reduces to a simple one, that is,

$$\left\langle\frac{W^2}{z+W^s}\right\rangle=W_c\frac{\Gamma(\mu+2)}{\Gamma(\mu)}\int_0^\infty dx\,\frac{e^{-zx/W_c}}{(1+x)^{Z\mu-\mu+2}} \tag{A4}$$

In the same way one has

$$\left\langle \frac{WW'}{z+W^s} \right\rangle \equiv \left\langle \frac{W_i W_j}{z+W_1+\cdots+W_{Z-1}} \right\rangle$$
$$= \frac{W_c}{[\Gamma(\mu)]^{Z-1}} \int_0^\infty \cdots \int_0^\infty dx_1 \cdots dx_{Z-1}$$
$$\times \frac{x_i x_j x_1^{\mu-1} \cdots x_{Z-1}^{\mu-1}}{z/W_c + x_1 + \cdots + x_{Z-1}} e^{-(x_1+\cdots+x_{Z-1})} \tag{A5}$$

This multiple integral also reduces to a simple one, that is,

$$\left\langle \frac{WW'}{z+W^s} \right\rangle = W_c \frac{[\Gamma(\mu+1)]^2}{[\Gamma(\mu)]^2} \int_0^\infty dx \frac{e^{-zx/W_c}}{(1+x)^{Z\mu-\mu+2}} \tag{A6}$$

Therefore

$$\left\langle \frac{W^2}{z+W^s} \right\rangle = \frac{\mu+1}{\mu} \left\langle \frac{WW'}{z+W^s} \right\rangle \tag{A7}$$

hence the desired result [Eq. (61) of the main text] by using Eqs. (A1) and (A2).

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